November 27, 2012

Crystalline Structure of Accretion Disks: Features of the Global Model

Giovanni Montani[1, 2, 3, *] and Riccardo Benini[2, †]
[1] *ENEA - C.R. Frascati, U.T.Fus. (FUSMAG Lab), Via Enrico Fermi, 45 (00044), Frascati (Roma), Italy*
[2] *Dipartimento di Fisica Università di Roma "Sapienza" Piazzale Aldo Moro 5, 00185, Roma, Italy*
[3] *INFN - Istituto Nazionale di Fisica Nucleare, Sezione di Roma 1*

In this paper, we develop the analysis of a two-dimensional magnetohydrodynamical configuration for an axially symmetric and rotating plasma (embedded in a dipole like magnetic field), modeling the structure of a thin accretion disk around a compact astrophysical object. Our study investigates the global profile of the disk plasma, in order to fix the conditions for the existence of a crystalline morphology and ring sequence, as outlined by the local analysis pursued in [1, 2]. In the linear regime, when the electromagnetic back-reaction of the plasma is small enough, we show the existence of an oscillating radial behavior for the flux surface function which very closely resembles the one outlined in the local model, apart from a radial modulation of the amplitude. In the opposite limit, corresponding to a dominant back-reaction in the magnetic structure over the field of central object, we can recognize the existence of a ring-like decomposition of the disk, according to the same modulation of the magnetic flux surface, and a smoother radial decay of the disk density, with respect to the linear case. In this extreme non-linear regime, the global model seems to predict a configuration very close to that of the local analysis, but here the thermostatic pressure, crucial for the equilibrium setting, is also radially modulated.

Among the conditions requested for the validity of such a global model, the confinement of the radial coordinate within a given value sensitive to the disk temperature and to the mass of the central objet, stands; however, this condition corresponds to dealing with a thin disk configuration.

PACS numbers: 97.10.Gz, 52.30.-q, 52.30.Cv

* giovanni.montani@frascati.enea.it
† riccardo.benini@gmail.com

I. INTRODUCTION

The understanding of the accretion process [3] characterizing compact astrophysical objects is regarded as a central issue when describing the triggering of high energy phenomena, such as the formation of jets. Indeed, it is commonly believed that the intense electromagnetic emission in different bands from astrophysical sources, such as Gamma Ray Bursts (see for example [4]) or Active Galactic Nuclei (see for example [5]), should be the final result of the proper instabilities of the accretion mechanism. Theoretical attempts have been made over the years to explain the formation of jets from the accretion profile of compact and massive astrophysical systems (see, among others, [6] and [7]).

The accretion mechanism in astrophysical systems is, however, far from being settled, because many aspects of the angular momentum transport within the gravitating plasma are still open to scientific debate. The stellar disk configurations, on which the present analysis is focused, are described by a well-established paradigm, which, for the case of a thin disk profile, has the character of a standard model [3]. This scenario satisfactorily reproduces the phenomenology fixed by the observational data, but at the price of postulating a viscoresistive magnetohydrodynamic (MHD) approach, which is not supported by the microscopic features of the disk plasma. The possibility to reconcile such an effective description of the non ideal nature of the plasma with the microscopic estimations of the viscosity and resistivity coefficients is individuated in the appearance of strong turbulence behaviors inside the disk. The theoretical framework for describing the onset of the plasma instabilities is offered by the so-called magnetorotational instability, introduced in [8, 9] (see also [10, 11]).

Even if the standard model for the thin accretion disks is rather successful and widely accepted, it cannot escape the following two relevant criticisms. (i) The z-dependence of the configuration is somewhat frozen out from the equilibrium by an average procedure along the vertical direction of both the radial and azimuthal equations, as originally proposed in [12, 13]. (ii) The experience acquired in plasma morphology from the laboratory activities, does not confirm the direct relation between the existence of a turbulent regime and the applicability of the viscoresistive MHD scenario [14].

In [15] it is shown how the configuration of a thin disk, treated in a two-dimensional ideal MHD scheme, reveals very different features with respect to the effective one-dimensional standard model. In fact, a new equilibrium profile, corresponding to a strong magnetization of the disk, is determined by virtue of the confinement induced by the Lorentz force. The main significant property of this regime is the non-Keplerian character of the angular velocity of the disk.

A more specific criticism to the standard model is pursued in [1, 2], where a fundamental description of the rotating disk plasma is addressed by postulating the ideal two-dimensional MHD and arguing that the accretion mechanism can be driven by intermittent ballooning instabilities which push the plasma toward the central object via its porosity near the X-points of the magnetic configuration [16]. Until now, the main achievement of this new point of view, is the demonstration that the local configuration around a fixed value of the radial coordinate, outlines the decomposition of the disk into a sequence of rings, corresponding to adjacent and opposite current density filaments. This issue takes place only in the limit of a very strong electromagnetic back-reaction of the plasma (its existence in the limit of large values of the plasma β parameter has been investigated in [17]), but a periodic (crystalline) structure of the magnetic surface functions already emerges in the linear regime.

In the present analysis, we fix the conditions necessary to recover the crystalline and ring like profile within a *global* two-dimensional MHD model. In other words, we quantify the role played by the radial envelope in determining the amplitude scaling of the oscillating behavior that characterizes the magnetic configuration.

In the linear case, we are able to sketch a clear picture for the emergence of the crystalline profile from the disk configuration, establishing that the amplitude of the perturbed magnetic surface scales as $\sqrt{r}$, while the mass density on the equatorial plane decays as r^{-3} (r being the radial coordinate). In order to obtain this scheme from the generic axially symmetric MHD scenario, we need to make some significant assumptions, which, however, can be mainly summarized by the thin nature of the disk, the short scale of the perturbations in the plasma, and by the confinement of the configuration well-inside a given radial region, determined by the model parameters (such as the mass of the central object and the disk temperature).

When the electromagnetic back-reaction within the disk plasma induces a dominant magnetic field (with respect to the one provided by the central object), we recover the same ring like decomposition described in the local model [2]. The radial modulation of the amplitude of the oscillations of the magnetic surfaces is the same as in the linear case, but now the mass density of the disk decays slower along the radial direction, i.e., like $r^{-1/2}$ instead of r^{-3}. Furthermore, in the non linear case, the pressure assumes a significant role in fixing the equilibrium, and its oscillation amplitude is also modulated as r^{-1}, while retaining the same structure as in the local model.

The main achievement of the present analysis is to show that the local features outlined in [1] for the steady configuration of a thin disk have a global character too and therefore they may concern fundamental astrophysical processes, taking place within the disk plasma, such as the realization of material jets [18, 19]. Furthermore,

we provide a precise constraint about the internal radial region in which the crystal profile of the magnetic field can take place, identifying a parameter discriminating for specific astrophysical sources.

The paper is organized as follows. In Sec. II, we give the motivation for a reformulation of the accretion problem toward a more ideal nature of the gravitating plasma. In Sec. III, we review the fundamental equations of axially-symmetric two-dimensional MHD describing an accretion disk embedded in the gravitational and magnetic fields of a central object; furthermore we fix the relation between the angular frequency of the plasma and the flux function. In Sec. IV, we develop the perturbation scheme for the considered problem, expanding the configuration equations up to the first order in agreement with the assumption of dealing with small-scale perturbations, and fix the conditions needed to recover the crystalline structure in the global case. In Sec. V, we assess the linear case when the back-reaction of the plasma is much smaller than the dipole-like field of the central object, and, in Sec. VI, we discuss the phenomenological implications of the constructed model in order to outline the main features of the global plasma configuration. Then, in Sec. VII, we analyze the opposite case, i.e., when the back-reaction of the plasma cannot be considered as a small perturbation but it is the dominant contribution in fixing the global profile. Concluding remarks follow in Sec. VIII.

II. NECESSITY FOR A REFORMULATION OF THE ACCRETION PICTURE

In the *Standard Model* for the stellar accretion [3, 12], the configuration of an axisymmetric thin disk is determined by the fluidodynamical equilibrium which takes place in the gravitational field of the compact accreting object (having mass M_*), but performing an average procedure along the vertical direction.

The radial equilibrium states that the angular frequency of the disk takes the Keplerian profile $\omega(r) = \omega_{\rm K} = \sqrt{GM_*/r^3}$. Significant deviations from such a behavior are expected only in advective dominated regimes.

The vertical equilibrium in the isothermal disk of temperature T, fixes the exponential decay of the mass density ρ over the equatorial plane value $\rho_0(r)$, i.e., $\rho/\rho_0 \equiv D(z^2) = \exp\{-z^2/H^2\}$, where we introduced the typical length $H^2 = 2v_{\rm s}^2/\omega_{\rm K}^2$ estimating the half-depth of the disk ($v_{\rm s}$ is the sound velocity on the equatorial plane, namely $v_{\rm s}^2 = 2K_{\rm B}T/m_{\rm i}$, with $K_{\rm B}$ the Boltzmann constant and $m_{\rm i}$ the ion mass). The azimuthal equilibrium describes the angular momentum transport across the disk, by virtue of a turbulent viscosity coefficient $\mathbb{D}$, such that

$$\dot{M}_{\rm d}(L - L_{\rm d}) = 3\pi\mathbb{D}\omega_{\rm K}r^2\,, \qquad (1)$$

where L is the angular momentum per unit mass, $L_{\rm d}$ is a fixed value and $\dot{M}_{\rm d} = -2\pi r\Sigma v_r$ is the mass accretion rate, associated with the radial equatorial velocity $v_r < 0$ and the surface mass density $\Sigma \equiv \int_{-H}^{H}\rho dz$. Finally, the continuity equation implies that $\dot{M}_{\rm d} = \mathrm{const} > 0$ (a discussion of these equations can be found, for example, in [3]).

A. The viscoresistive puzzle

The microscopic plasma structure accounts for a too small viscosity coefficient $\mathbb{D}$ arising in the disk to explain the accretion rates observed in some astrophysical systems, such as X-ray binaries. In fact, the observed accretion rates, evaluated by the increasing disk luminosity $\dot{L}_{\rm d} \sim GM_*\dot{M}_{\rm d}/R_*$, require large values of $\mathbb{D}$, that in [12] were associated with a postulated turbulent behavior of the disk fluid. Since by definition $L = \omega r^2$, we can infer $\mathbb{D} = 2\Sigma v_{\rm t}H/3$, $v_{\rm t}$ being a turbulence velocity, expressible as $v_{\rm t} = \alpha v_{\rm s}$, where α is a free parameter. The fundamental question is whether the axisymmetric disk is linearly stable with respect to small perturbations that preserve its symmetry. A solution to this problem comes from the presence of a non-vanishing magnetic field, which makes the non-linear interaction of very small disturbances of the equilibrium possible. Such a MHD instability, commonly known as *Magneto-Rotational Instability* (MRI), is triggered by the radial gradient of the disk angular velocity and has been fixed by the Velikov analysis of 1959 [8, 9] (for a review on the topic, see [11]). In [20] it is argued that the MRI is strongly suppressed when the disk is thin enough and therefore its efficiency in generating turbulence is ruled out in favor of a *thermorotational* instability, in which the vertical gradient of the temperature plays a crucial role.

Indeed, the powerful scenario emerging from the MRI relies on the presence of an even small, embedded magnetic field. However, the existence of an unstable mode of wavenumber k is subjected to the condition

$$k^2v_{\rm A}^2 + \frac{d\omega^2}{d\ln r} \leq 0\,, \qquad \omega(r) \simeq \omega_{\rm K} = \sqrt{\frac{GM_*}{r^3}}\,, \qquad (2)$$

where $v_{\rm A}$ denotes the background Alfvén velocity. This simple condition can be easily recovered when the space dependence of the perturbation is parallel to the direction of magnetic field, here assumed without a significant loss of generality, along the z-axis. Condition (2) can be recast in a more significant shape, as far as we realize that the stability of the disk profile as a whole requires the extension of such inequality to scale of the disk depth. A global disk stability is, in fact, ensured only if the Alfvén term dominates even for the smallest available wavenumbers $k \simeq \pi\omega_{\rm K}/\sqrt{2}v_{\rm s}$. Hence, observing that $d\omega_{\rm K}^2/d\ln r = -3\omega_{\rm K}^2$, we finally get

$$v_{\rm A}^2 \leq \frac{6}{\pi^2}v_{\rm s}^2\,. \qquad (3)$$

As far as the magnetic field being sufficiently small and the disk thick enough, the emergence of an unstable mode

is guaranteed. However, if the magnetic field of the central object is important, so that the Alfvén velocity is not too small and at the same time, the temperature and angular rotation of the disk are able to produce a sufficiently thin profile (for which the sound speed is constrained by a small upper bound $v_{\rm s}/(\omega_{\rm K} r) \sim H/r \ll 1$), behaviors violating condition (3) must be taken into account. In other words, we cannot exclude the existence of a class of thin disks for which the plasma admits a parameter β less than a few units. Indeed, in the isothermal case we have

$$\beta \equiv 2\frac{v_{\rm s}^2}{v_{\rm A}^2} \leq \frac{\pi^2}{3} \, . \tag{4}$$

For such relatively cold and strongly magnetized plasma disks, the MRI is suppressed and we need a new type of instability to account for the turbulent scenario required to deal with significant dissipative effects. Interesting features in favor of new instability perspectives are discussed in [20–22].

The presence of an intense magnetic field (as required by a pulsar-like central object) however, poses a new puzzle regarding the value of the non-zero resistivity coefficient η of the ideal plasma. In fact, the equation of the electron force balance reads as

$$\vec{E} + \frac{\vec{v}}{c} \wedge \vec{B} = \eta \vec{J} \, , \tag{5}$$

with $\vec{E}$ and $\vec{B}$ denoting the electric and magnetic field, respectively, while $\vec{v}$ is the plasma velocity and $\vec{J}$ is the current density. Since the axial symmetry requires $E_\phi \equiv 0$, the azimuthal component stands as

$$v_z B_r - v_r B_z = c \eta J_\phi \, . \tag{6}$$

In the one-dimensional model, obtained for the thin disk by averaging along the z-axis, and taking into account both the relation between v_r and the constant accretion rate $\dot{M}_{\rm d}$, as well as the dipole-like morphology of the magnetic field, Eq.(6) is given by

$$\frac{\dot{M}_{\rm d}\mu_0}{2\pi\Sigma(r) r^4} = \eta c J_\phi \, , \tag{7}$$

μ_0 being a parameter modulating the intensity of the dipole field. Observing that the superficial density is given by [3]

$$\Sigma(r) = \left(\frac{2\pi v_{\rm s}^2}{GM_*}\right)^{1/2} \rho_0(r) r^{3/2} \, , \tag{8}$$

the azimuthal current density J_ϕ takes the radial behavior

$$J_\phi = \frac{\dot{M}_{\rm d}\mu_0}{\eta c v_{\rm s}} \sqrt{\frac{GM_*}{(2\pi)^3}} \frac{1}{\rho_0(r) r^{11/2}} \, . \tag{9}$$

The puzzle consists of the fact that the strong dipole magnetic field is associated with a zero density current (being a vacuum solution), and thus the toroidal current J_ϕ must be rather small since its existence is due to the plasma backreaction only. The key quantity we have to focus on is therefore the ratio $\dot{M}_{\rm d}\mu_0/\eta$, which is requested to be very small as well. For an X-ray star, for which the parameters $\dot{M}_{\rm d}$ and μ_0, as determined from the observations, are significantly high, the smallness of the toroidal current J_ϕ can be reached only for a sufficiently high value of the resistivity coefficient. But this is clearly not the case, as far as we microscopically estimate this coefficient, accordingly to the expression (in Gauss CGS units, i.e., in seconds)

$$\eta = \frac{m_{\rm e}\nu_{\rm pe}}{n_{\rm e} e^2} \, , \tag{10}$$

where $m_{\rm e}$ is the electron mass, $\nu_{\rm pe}$ is the proton-electron collision frequency, $n_{\rm e}$ is the electron number density, and e is the electron charge. The proton-electron collision frequency $\nu_{\rm pe}$ can be well approximated through the electron-electron collision frequency $\nu_{\rm ee}$, given by[1]

$$\nu_{\rm ee} = 2.91 \times 10^{-6} \left(\frac{n_{\rm e}}{1\,{\rm cm}^{-3}}\right)\left(\frac{T}{1\,{\rm eV}}\right)^{-\frac{3}{2}} \log(\Lambda) {\rm Hz} \, , \tag{11}$$

where the Coulomb logarithm $\log(\Lambda)$ can be estimated via the formula

$$\Lambda = 12\pi N_{\rm D} \, , \tag{12}$$

$N_{\rm D}$ being the Debye number of the plasma. In Fig.1 this resistivity coefficient is plotted against the number density and the temperature of the plasma, showing the smallness of the corresponding assumed values.

[1] see the *NRL Plasma Formulary* (Naval Research Laboratory, Washington, D.C., 2009).

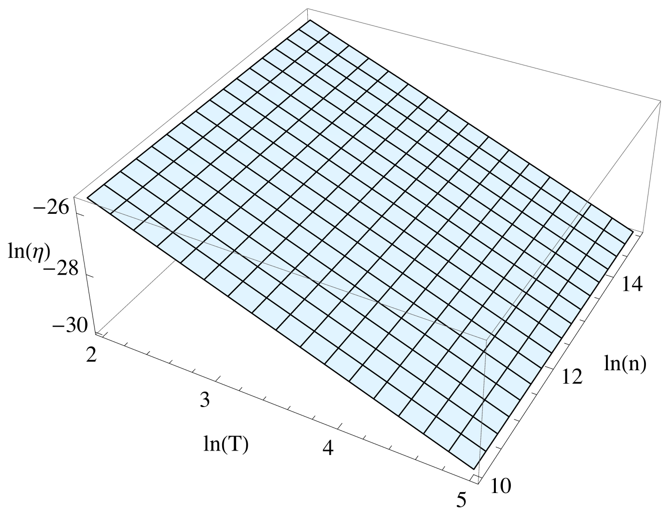

Figure 1. Resistivity coefficient η as a function of the electron number density n_e (cm^{-3}) and of the temperature of the plasma T (eV).

The viscoresistive puzzle is therefore the combination of two related facts: a magnetic field is required by the MRI, which allows the onset of turbulence, to generate the viscosity term (without such dissipative effect no angular momentum transport is permitted); this magnetic field in turn requires an anomalous resistivity to get a satisfactory balance for the electron force along the azimuthal component. We are led, thus, to postulate both a high viscosity and resistivity coefficients, i.e., a magnetic Prandtl number of order unity [7]. However, no firm explanation exists for the possibility that the turbulent behavior, resulting from the MRI, is able to generate significant dissipative effects in a thin disk, and, as discussed above, some plasma configurations can exist for which the MRI does not work at all.

B. The crystalline structure: from the local toward a global model

This apparent inconsistency of the standard model for stellar accretion (at least for sufficiently magnetized astrophysical sources) and the possible failure of the MRI for a sufficiently thin disk, led Coppi in [1, 2] to reformulate the thin disk equilibrium in a pure two-dimensional ideal MHD scenario. The point of view proposed corresponds to replacing the diffusive morphology of the magnetic field in the plasma, resulting from the effective dissipative processes, by rigid structures in the magnetic flux surfaces and in the mass density profile, the so-called *crystalline structure*, with the associated ring sequence decomposition of the disk. In this scenario, the angular momentum transport is not ensured by the turbulent viscosity effect, but is argued to come from the porosity of the plasma near the X-points of the magnetic configuration, where the z-component of the $\vec{B}$ field vanishes. Indeed, near such peculiar configurations, the radial velocity can increase without any violation of the electron force balance. In such a framework, the plasma is pushed to infall by the action of intermittent thermorotational unstable modes, similar to the *ballooning* modes observed in laboratory experiments [23, 24]; for a discussion of this inferred alternative scenario for the disk accretion, see [16].

We also stress how the possibility to reconcile the crystalline structure with the viscoresistive MHD scenario is discussed in [25], showing how the radial oscillation of the magnetic flux surfaces can be recognized, for a local model, even in the presence of dissipative effects. Furthermore, [26] studied the ideal MHD configuration of a thin disk near a fixed radius from the central object, but differently from the original works [1, 2], the contribution of small poloidal currents and matter fluxes are retained in the problem. The idea proposed concerns the possibility to balance the electron force equation by requiring the peculiar condition $v_zB_r - v_rB_z \simeq 0$ (to be intended as a net average prescription). In the framework of a plasma decomposed in a ring sequence, the possibility to deal with strong values of the vertical velocity, in correspondence to suitable restrictions, is shown. The establishment of such configurations seems to be of relevance in the explanation of stellar wind profiles, as well as in the determination of seeds for the jet formation.

All these results on the crystalline structure, however, are derived within the scheme of a local model, in which the plasma configuration is fixed around a fiducial radius from the central object. The main goal of our analysis is the extension of the original local model to the global configuration of the disk.

Indeed, the local analysis is developed around a fixed value of the radial coordinate in the disk, and therefore it is unable to clarify how the differentially rotating layers determine the radial behavior of the fundamental physical quantities, such as the pressure, the mass density and the magnetic flux surfaces. One of the main questions left open is whether the crystalline structure and the ring sequence arising in the disk, are compatible with a regular radial envelop, i.e., if these peculiar features can be reconciled with a physically meaningful behavior of the fundamental configuration variables. For instance, an important point to be addressed, and successfully handled in this paper, is the possibility to have a radial oscillation of the mass density (characterizing the ring formation) but in the framework of a global decay of the radial matter distribution as the outer regions of the disk are approached.

Having this scenario in mind, we formulate the global disk plasma equilibrium starting from those natural assumptions which are the expected generalization of what was proposed in [1, 2]. As first step, we need to fix the explicit dependence that the plasma angular frequency outlines, according to the corotation theorem [27], with respect to the background magnetic surface; the latter,

in particular, are taken in the suitable form of a dipolar-like configuration. In the local model, this information is somehow hidden in a constant term and ultimately restated as the derivative of the angular frequency with respect to the radial variable evaluated at the fixed point. Here we fix the dependence $\omega = \omega(\psi)$ by determining it on the equatorial plane of the disk and then extending this relation to the whole plasma profile.

Another basic assumption at the ground of the global analysis presented below, is the request of dealing with a very small-scale backreaction of the plasma with respect to the magnetic field of the central object. In other words, we assume that the physical quantities (i.e., the internal currents, the mass density, pressure perturbations and the correction to the magnetic surfaces) have a very rapid radial variation, in agreement with the MHD. This request is equivalent to fixing a hierarchy in the radial gradient profile, and has the same impact on the plasma configuration as the smoothness of the plasma around the fixed radius, which is implicitly postulated in [1].

These two requirements at the ground of the global model, i.e., the dipole like nature of the central object magnetic field and the very small scale of the plasma backreaction, appear as very natural choices for the physical characterization of a stellar disk, and their successful implementation to obtain the crystalline array of the plasma along the radial direction, must be accompanied only by the restriction to deal with a thin disk; such an assumption (already present in the local analysis) is allowed by the many observations that such a class of disks is indeed present around compact and rapidly rotating astrophysical objects.

The real new contribution of what is discussed here, is the determination of the radial profile of the disk and the estimation of the microscopic nature (for an astrophysical setting) of the wavelength of the perturbations responsible for the ring decomposition of the disk. Such an analysis has no direct relation with [25], where the poloidal currents and matter fluxes are taken into account; however, the natural development of the present global model would be to verify the existence of such peaks in the vertical velocity, once the radial envelope is fully accounted for. This scenario would allow the exact determination of the disk regions where the stellar wind or the jets could take place. Nonetheless, involving poloidal currents and matter fluxes in a global model would lead to non-trivial questions concerning the validity of the corotation theorem, well beyond the scope of the present approach. In fact, in the addressed toroidal picture, the validity of such a theorem stands as well and we are focusing our attention on the radial dependence of the crystalline structure, i.e., on how the magnetic flux surfaces and the mass density oscillate across the global radial profile of the disk.

III. FUNDAMENTAL EQUATIONS

We now fix the fundamental configuration equations for the axial symmetry of the disk, by considering the central astrophysical object as a star of mass M_*, endowed with an intense magnetic field $\vec{B}$. The magnetic field can be taken in the form

$$\vec{B} = -\frac{1}{r}\partial_z\psi\hat{e}_r + \frac{1}{r}\partial_r\psi\hat{e}_z\,, \tag{13}$$

$\psi = \psi(r, z^2)$ being the magnetic flux function, while the associated current density $\vec{J}$ remains fixed as

$$\vec{J} = -\frac{c}{4\pi}\left[\partial_r\left(\frac{1}{r}\partial_r\psi\right) + \frac{1}{r}\partial_z^2\psi\right]\hat{e}_\phi\,. \tag{14}$$

The magnetic field of the central object is well described in the disk by a dipole-like configuration, corresponding to the flux function

$$\psi_{\mathrm{D}}(r, z^2) = \frac{\mu_0 r^2}{(r^2+z^2)^{3/2}}\,, \qquad \mu_0 = \mathrm{const}\,. \tag{15}$$

Here, the constant value μ_0 fixes the dipole field amplitude and we stress that the current density associated with this vacuum configuration is characterized by the relation $\vec{J}(\psi = \psi_{\mathrm{D}}) \equiv 0$.

The Newton potential χ describing the gravitational field generated by the central object stands as

$$\chi(r, z^2) = \frac{GM_*}{\sqrt{r^2+z^2}}\,, \tag{16}$$

G being the Newton constant. We can also define the Keplerian angular velocity ω_{K} as

$$\omega_{\mathrm{K}}^2(r, z^2) = \frac{GM_*}{(r^2+z^2)^{3/2}}\,. \tag{17}$$

We observe that on the equatorial plane $z = 0$, the following relations hold

$$\left.\begin{aligned} \omega_{\mathrm{K}}^2(r,0) \equiv \omega_{\mathrm{K}0}^2 &= \frac{GM_*}{r^3} \\ \psi_{\mathrm{D}} &= \frac{\mu_0}{r} \end{aligned}\right\} \;\Rightarrow\; \omega_{\mathrm{K}0}^2 = \frac{GM_*}{\mu_0^3}\psi_{\mathrm{D}}^3\,. \tag{18}$$

These relations are of interest because of the thin nature of the disk. Furthermore the corotation theorem [27] states that the plasma angular velocity must be a function of the magnetic surface only, i.e., $\omega = \omega(\psi)$, and it is natural to postulate that the relation

$$\omega^2 = \frac{GM_*}{\mu_0^3}\psi^3\,, \tag{19}$$

holding on the equatorial plane, is valid everywhere in the disk. We note that, in agreement with Eq.(17), the disk, embedded in the dipole magnetic field of the central

object, can not have a Keplerian behavior far from the plane $z = 0$.

The radial equilibrium of the disk configuration corresponds to the following force balance

$$GM_*\rho\left[-\frac{r\psi^3}{\mu_0^3}+\frac{\psi_{\rm D}}{\mu_0 r}\right]=-\partial_r p-\frac{\partial_r\psi}{4\pi r}\mathcal{D}[\psi]\,, \qquad (20)$$

where $\rho = \rho(r, z^2)$ and $p = p(r, z^2)$ denote the mass density and the thermostatic pressure, respectively.

The vertical configuration equation reads as

$$-\partial_z p-\rho\frac{GM_* z}{\mu_0 r^2}\psi_{\rm D}-\frac{\partial_z\psi}{4\pi r}\mathcal{D}[\psi]=0\,, \qquad (21)$$

where

$$\mathcal{D}[\psi]\equiv\partial_r\left(\frac{1}{r}\partial_r\psi\right)+\frac{1}{r}\partial_z^2\psi\,. \qquad (22)$$

IV. PERTURBATION SCHEME

We now split the flux surface function ψ as follows:

$$\psi=\psi_{\rm D}+\zeta,\qquad \mid\zeta\mid\ll\mid\psi_{\rm D}\mid\ . \qquad (23)$$

Here $\zeta(r, z^2)$ describes the electromagnetic backreaction of the confined plasma. Via the approximation $\psi^3 \simeq \psi_{\rm D}^3 + 3\psi_{\rm D}^2\zeta$, and recalling that the toroidal currents, associated with $\psi_{\rm D}$ identically vanish, the radial and vertical configuration equations rewrite as

$$\begin{aligned}&\rho\frac{GM_*}{\mu_0}\left[-\frac{r}{\mu_0^2}\left(\psi_{\rm D}^3+3\psi_{\rm D}^2\zeta\right)+\frac{\psi_{\rm D}}{r}\right]\\&=-\partial_r p-\frac{1}{4\pi r}\left(\partial_r\psi_{\rm D}+\partial_r\zeta\right)\mathcal{D}[\zeta]\,,\end{aligned} \qquad (24a)$$

$$\partial_z p+\rho\frac{GM_* z}{\mu_0 r^2}\psi_{\rm D}+\frac{1}{4\pi r}\left(\partial_z\psi_{\rm D}+\partial_z\zeta\right)\mathcal{D}[\zeta]=0\,, \qquad (24b)$$

respectively. In these equations, we retained the gradients of ζ, because, in agreement with the *drift ordering*, they can be relevant and even dominant, despite the smallness of ζ.

Taking into account the thin nature of the disk, i.e., that its half-depth $H(r)$ satisfies the relation $H(r) \ll r$, we can expand the vertical dependence of the above equations in agreement with the approximation $(r^2 + z^2)^a \simeq r^{2a}(1 + az^2/r^2)$. Hence, the configurational system Eq.(24) is recast as

$$\begin{aligned}&\frac{3\rho GM_*}{r^4}\left(z^2-\frac{(r^3-3rz^2)\zeta}{\mu_0}\right)=\\&=-\partial_r p-\frac{1}{4\pi r}\left[-\frac{\mu_0}{r^2}\left(1-\frac{9z^2}{2r^2}\right)+\partial_r\zeta\right]\mathcal{D}[\zeta]\,,\end{aligned} \qquad (25a)$$

$$\partial_z p+\rho\frac{GM_* z}{r^3}+\frac{1}{4\pi r}\left(-\frac{3\mu_0 z}{r^3}+\partial_z\zeta\right)\mathcal{D}[\zeta]=0\,. \qquad (25b)$$

We now split the mass density ρ and the pressure p into the background (overbar) and perturbation (careted) components, as follows

$$\rho=\bar\rho+\hat\rho\,,\qquad p=\bar p+\hat p\,, \qquad (26)$$

requiring that the quantities $\bar\rho$ and $\bar p$ are linked via the isothermal relation $\bar p = v_{\rm s}^2\bar\rho$. We also determine the form of $\bar\rho(r, z^2)$ by imposing the validity of the gravothermal vertical equilibrium

$$\partial_z\bar p+\bar\rho\omega_{\rm K0}^2 z=0\ \Rightarrow\ \bar\rho=\rho_0(r)\exp\left(-\frac{\omega_{\rm K0}^2 z^2}{2v_{\rm s}^2}\right)\,, \qquad (27)$$

where we remind that $\rho_0(r)$ denotes the mass density on the equatorial plane. Observing that

$$\partial_r\bar p=\left(\frac{3GM_* z^2}{2r^4}+\frac{v_{\rm s}^2}{\rho_0}\frac{d\rho_0}{dr}\right)\bar\rho\,, \qquad (28)$$

in order to restate the radial and the vertical equilibria in a simpler form, we are led to require that the following conditions hold

$$\frac{\zeta r}{\mu_0}=\frac{\zeta}{\psi_{\rm D0}}\gg\frac{3z^2}{2r^2}\,,\quad\frac{z^2}{r^2}\ll1\,,\quad\frac{\zeta}{\psi_{\rm D0}}\gg\gamma\frac{rv_{\rm s}^2}{3GM_*}\,, \qquad (29)$$

($\psi_{\rm D0} = \psi_{\rm D}(r, z = 0)$). Here we make the ansatz $\rho_0 \propto r^\gamma, \gamma = \text{const}$ (see below) and obtain the following system of equations from Eqs.(25)

$$\begin{aligned}&(\bar\rho+\hat\rho)\,\frac{3GM_*}{\mu_0 r}\left(1-3\frac{z^2}{r^2}\right)\zeta\\&=\partial_r\hat p+\frac{1}{4\pi r}\left[-\frac{\mu_0}{r^2}\left(1-\frac{9z^2}{2r^2}\right)+\partial_r\zeta\right]\mathcal{D}[\zeta]\,,\end{aligned} \qquad (30a)$$

$$\partial_z\hat p+\hat\rho\frac{GM_* z}{r^3}+\frac{1}{4\pi r}\left(-\frac{3\mu_0 z}{r^3}+\partial_z\zeta\right)\mathcal{D}[\zeta]=0\,. \qquad (30b)$$

A. Toward the crystalline structure

In order to find the region of applicability for the crystalline structure of the disk, outlined in the local model of Coppi [1], we impose some specific restrictions. Denoting by k the wavenumber of the radial dependence characterizing $\phi = \zeta/\sqrt{r}$, we make the request to live in the disk zone where $kr \gg 1$. Thus, under such a restriction and neglecting the quantity $\partial_z\psi_{\rm D}$ in the vertical force balance, the disk configuration is determined via the following system

$$\begin{aligned}&\frac{3GM_*\rho_0(r)}{\mu_0\sqrt{r}}\left(\exp\left\{-\frac{\omega_{\rm K0}^2 z^2}{2v_{\rm s}^2}\right\}+\hat D\right)\left(1-3\frac{z^2}{r^2}\right)\phi\\&=\partial_r\hat p+\frac{1}{4\pi r}\left[\partial_r\phi-\frac{\mu_0}{r^{5/2}}\left(1-\frac{9z^2}{2r^2}\right)\right]\left(\partial_r^2\phi+\partial_z^2\phi\right)\,,\end{aligned} \qquad (31a)$$

$$\partial_z \hat{p} + \hat{\rho}\frac{GM_* z}{r^3} + \frac{1}{4\pi r}\partial_z\phi\left(\partial_r^2\phi + \partial_z^2\phi\right) = 0\,, \quad (31b)$$

where we defined $\hat{D} = \hat{D}(r, z^2) \equiv \hat{\rho}/\rho_0$. If we indicate by h the characteristic scale for the z-dependence of ζ, the possibility to neglect the term $\partial_z\psi_{\rm D}$ in the vertical equilibrium, relies on the validity of the requirement $\zeta/\psi_{\rm D} \gg 3zh/r^2$.

V. THE LINEAR CASE

We now study the linear case, corresponding to a sufficiently low electromagnetic backreaction in the plasma, so that we can neglect non-linear terms in ϕ when fixing the equilibrium. For instance, $\partial_r\psi_{\rm D} \gg \partial_r\zeta$ is equivalent to the condition

$$\frac{\zeta}{\psi_{\rm D}} \ll \frac{1}{kr}\,. \quad (32)$$

Furthermore, we observe that the background mass density can be expanded as

$$\bar{\rho} \simeq \rho_0(r)\left(1 - \frac{GM_* z^2}{2r^3 v_{\rm s}^2}\right) = \rho_0(r)\left[1 - \left(\frac{R_{\rm S}c^2}{4rv_{\rm s}^2}\right)\frac{z^2}{r^2}\right]\,, \quad (33)$$

where $R_{\rm S} \equiv 2GM_*/c^2$ is the Schrwarzshild radius of the central object. In what follows, we assume to be in that region of the disk where the condition

$$\frac{R_{\rm S}}{4r} \gg \frac{v_{\rm s}^2}{c^2}\,, \quad (34)$$

holds and we can approximate, in this linear regime, the configurational system (31) as

$$\rho_0\frac{3GM_*}{\mu_0}\left(1 - \frac{L_{\rm s}z^2}{r^3}\right)\phi + \frac{\mu_0}{4\pi r^3}\left(\partial_r^2\phi + \partial_z^2\phi\right) = 0\,, \quad (35a)$$

$$\partial_z\hat{p} + \hat{\rho}\frac{GM_* z}{r^3} = 0\,, \quad (35b)$$

where

$$L_{\rm s} \equiv GM_*/2v_{\rm s}^2\,, \quad (36)$$

(see Fig.2). In these equations we neglected $\hat{D}$, because $\bar{\rho} \gg \hat{\rho}$ in the linear regime, and also the radial pressure gradient, in comparison to the Lorentz force. We observe that condition (34) reads as $r \ll L_{\rm s}$ and that it is implied by condition (29). Equation (35b) tells us that the perturbations follow the background behavior, while the corresponding radial equilibrium (35a) has an intriguing feature: if we take $\rho_0 = m/r^3$, with $m =$ const. and

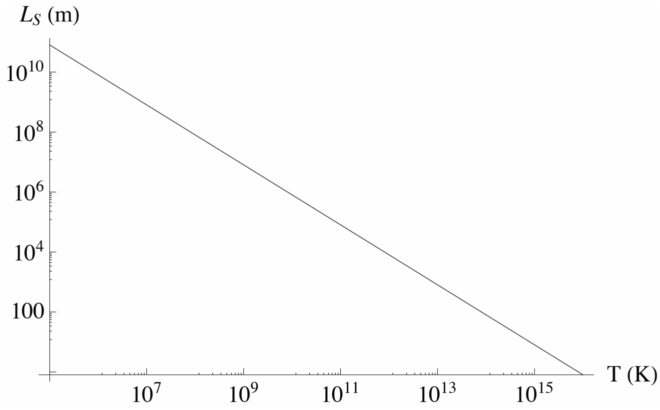

Figure 2. The parameter $L_{\rm s}$ as a function of the temperature T of the isothermal disk for a $M_* = 2M_\odot$

define $k^2 = 12\pi GM_* m/\mu_0^2$, then we arrive at the radial equilibrium equation in the form

$$\left(\partial_r^2\phi + \partial_z^2\phi\right) = -\left(1 - \frac{L_{\rm s}z^2}{r^3}\right)k^2\phi\,. \quad (37)$$

In the considered limits $z/r \ll 1$, $kr \gg 1$ and for $r \gg (L_{\rm s}/k^2)^{1/3}$ (and remembering $r \ll L_{\rm s}$), the solution of the equation above takes the form

$$\phi(r, z^2) = A\sin(kr)\exp\left\{-\frac{k\sqrt{L_{\rm s}}z^2}{2r^{3/2}}\right\}\,, \quad (38)$$

A being a constant amplitude. This oscillating form of ϕ, i.e., of $\zeta = \sqrt{r}\phi$ restores, under the set of conditions we fixed, the crystalline structure emerging in the local model of Coppi. The local oscillating behavior of ζ is sketched in Fig.3, while the global behavior is described in Fig.4.

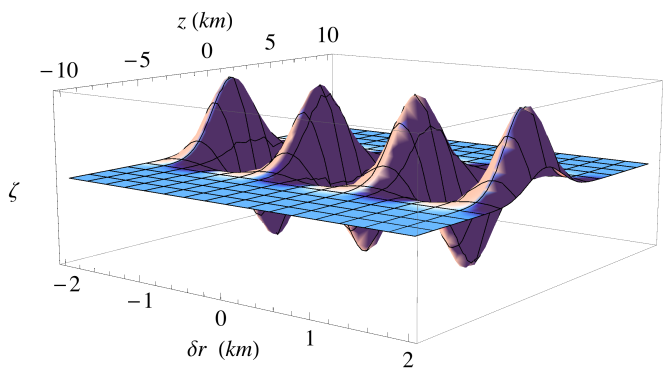

Figure 3. Local oscillating behavior of the perturbed flux function ζ around a radius $r = 10^3$km. The chosen parameters are: $A = 1$, $T = 10^7$K, $L_{\rm s} \simeq 8 \times 10^5$Km, $B = 10^{12}$Gauss and $m = 0.02M_\odot$

All the conditions imposed on the global model to recover the linearized radial oscillating behavior of the

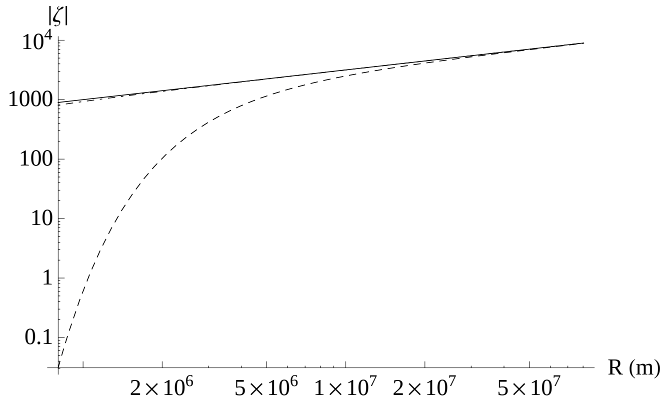

Figure 4. Global profile of the perturbed flux function ζ for fixed values of $z = 0, 10^2$ and 10^3m (clearly, the oscillating factor $\sin(kr)$ has been omitted). The chosen parameters are the same as in Fig.3. The weak radial dependence of the amplitude is evident.

magnetic surface function, can be summarized in the following three physical restrictions on the considered disk model:

1. The disk is assumed to be thin, i.e., $z^2/r^2 \ll 1$.

2. The wavelength of the perturbations, due to the electromagnetic backreaction, is much smaller than the disk size, i.e., $kr \gg 1$.

3. The disk configuration must be restricted to the radial region where the condition $r \ll L_{\rm s}$ holds, i.e.,

$$r \ll \frac{GM_*}{2v_{\rm s}^2} = \frac{R_{\rm S}c^2}{4v_{\rm s}^2} = \frac{m_{\rm i}c^2}{8K_{\rm B}T^2}R_{\rm S}\,, \tag{39}$$

where, the sound velocity $v_{\rm s}$ has ben expressed via the disk temperature T.

In this scheme, the existence of the crystalline structure requires that the disk temperature not exceed a given value. For instance, for a neutron star of about two Solar masses and a disk size of 10^6km, in order to apply our model over the whole disk, we would get the condition $K_{\rm B}T \ll 1$keV. Such an estimation leads us to infer that the present result, as that obtained in [1], remains valid for a class of rather cold disks.

VI. PHENOMENOLOGICAL IMPLICATIONS

Let us analyze the physical hints that we can get from the analysis of the global model for the crystalline structure, as traced above.

In order to compare the radial scaling of the perturbations to the size of the allowed region for the crystalline structure, we have to consider the ratio between the wavelength $\lambda = 2\pi/k$ and the typical scale $L_{\rm s}$, i.e.,

$$\frac{\lambda}{L_{\rm s}} = \sqrt{\frac{\pi\mu_0^2}{3GM_*m}\frac{2v_{\rm s}^2}{GM_*}} = \sqrt{\frac{32\pi}{3}}\frac{v_{\rm s}^2}{c^2}\sqrt{\frac{\ell^3}{R_{\rm S}^3}}\,, \tag{40}$$

where we defined the characteristic length

$$\ell \equiv \sqrt[3]{\frac{\mu_0^2}{mc^2}}\,. \tag{41}$$

Since $v_{\rm s} < c$ and $\ell \ll R_{\rm S}$ (for a typical neutron star we have value in the range $\sim$ 10m; see below for more details), the above ratio is many orders of magnitude smaller than unity. The oscillation scale of the perturbations, therefore, lives in the microscales of the system, very much below the characteristic length $L_{\rm s}$. The most relevant phenomenological implication of this issue, is in the impact that the crystalline structure can have on the global configuration of the disk. The very small characteristic length of the magnetic flux surface oscillations prevents the ring sequence emerging in the non-linear regime (see Sec. VII) from being on a macroscopic scale, and so able to account for certain discontinuous structures resulting from the observations of real astrophysical compact objects. Such kind of macroscopic structures can take place in this framework in limiting situations only, i.e., when the parameter m is particularly small and μ_0 is sufficiently large. However, this consideration does not affect the relevance of the existence of such microstructures toward the global equilibrium profile. In fact, this microscopic nature of the crystalline structure and of the ring sequence, plays a crucial role in fixing the fundamental instabilities characterizing the disk profile. In particular, the small-scale radial oscillations can trigger instabilities having very different morphologies with respect to the MRI, as already discussed for the local configuration in [20]. Thus, more than through direct observations in the optical or X-ray bands, we expect that the existence of such structures inside the disk can be revealed by specific features of the triggered turbulent regimes that can be unveiled in the resulting emission processes. The role that the microscopic crystalline structure can play in the onset and in the establishment of a turbulent scenario, can deeply influence the transport processes inside the disk plasma, especially in view of energy-momentum transfer from the micro- to the meso-scales of the system.

Another relevant phenomenological question concerns the restricted region where the crystalline profile and the fragmentation of the disk in a ring series can take place. Indeed, the condition $r \ll L_{\rm s}$ seems to confine the radial domain allowed for the magnetic flux surface oscillations in an internal portion of the thin disk. From a phenomenological point of view, this inequality is sensitive to the accretion system parameters, i.e., the mass of the central object and the temperature of the disk, so

identifying which fraction of the plasma extension [2] is involved in the crystalline profile. In particular, for configurations extended enough, having very large values of $R_{\rm ext}$ and whose plasma is significantly hot (so that the sound velocity is a few orders of magnitude smaller than the speed of light), the restriction above can limit the small scale radial oscillations to a very tiny portion of the accreting plasma. However, it is possible to characterize the class of disks to which such restrictions can be applied in a more simple and meaningful way, which is also consistent with the approximation scheme adopted above. In fact, the restriction $r \ll L_{\rm s}$ can be easily restated as follows

$$r \ll L_{\rm s} = \frac{GM_*}{2v_{\rm s}^2} = \frac{\omega_{\rm K}^2(r)}{2v_{\rm s}^2} r^3 = \frac{r^3}{H^2(r)}\,, \tag{42}$$

where we recall that $H(r)$ is the effective half-depth of the isothermal disk (see Eq.(27)). Thus, the restriction $r \ll L_{\rm s}$ is equivalent to the requirement that the disk has a real thin profile, i.e., $H(r) \ll r$. Realizing such an equivalence allows us to join together two of the fundamental constraints at the ground of our derivation of the global model, i.e., $r \ll L_{\rm s}$ and $z/r \ll 1$. In fact the former, being equivalent to $H(r) \ll r$ automatically ensures the validity of the latter. It is worth noting, however, that for a thick disk, while the constraint $r \ll L_{\rm s}$ is necessarily violated (with non-trivial implications for the existence of a global crystalline profile), the condition $z \ll r$ can still be satisfied in the proximity of the equatorial plane of the axisymmetric configuration.

We conclude this section by stressing how the two fundamental requirements we relied on when deriving the radial oscillation of the disk morphology, overlap the fundamental hypotheses of dealing with a thin disk in which the plasma backreaction is a small scale phenomenon, namely, $H(r) \ll r$ and $kr \gg 1$, respectively. This issue makes our analysis fully consistent with the ideas introduced in [1, 2] and substantiates the guess that the crystalline morphology and the ring sequence scenario are very general features of an accreting plasma well confined close to the equatorial plane. However, our study also has the merit to outline the microscopic nature of the backreaction scale (more than a simple small scale behavior postulated by Coppi), so opening a precise direction in the understanding of the role that such a radial periodicity of the plasma can play in the establishment of significant processes of transport of matter and angular momentum across the disk.

With respect to the estimation of the characteristic length ℓ, it is worth noting the following relation (valid in the linear case) between the total mass of the disk $M_{\rm d}$ and the parameter m entering the equatorial distribution $\rho_0(r) = m/r^3$, i.e.,

$$\begin{aligned} M_{\rm d} &= 2\pi m \int_{R_{\rm int}}^{R_{\rm ext}} \int_{-H(r)}^{H(r)} \frac{1}{r^2} e^{-z^2/H^2(r)} dz dr \\ &= 8\pi^{3/2} m \,\mathrm{erf}(1) \sqrt{\frac{K_{\rm B}T}{GM_* m_{\rm i}}} \left(R_{\rm ext}^{1/2} - R_{\rm int}^{1/2}\right) \\ &\simeq 1.8 \times 10^3 m \sqrt{\frac{K_{\rm B}T}{\rm keV}}\,. \end{aligned} \tag{43}$$

Here $\mathrm{erf}(x) = 2\pi^{-1/2} \int_0^x e^{-t^2}\, dt$ is the error function, $M_* = 2M_\odot$ is the mass of the central object, and the last equality stands for an inner and external disk radii of $R_{\rm int} = 10^3$km and $R_{\rm ext} = 10^6$km respectively. As a result we find that the parameter m explicitly depends on the temperature T and on the mass of the disk $M_{\rm d}$ via the relation

$$m = \frac{5.5 \times 10^{-5} M_{\rm d}}{\sqrt{\frac{K_{\rm B}T}{\rm keV}}}\,. \tag{44}$$

Assuming that $M_{\rm d}$ takes values in the range $[10^{-5}, 10^{-2}]M_\odot$, $K_{\rm B}T$ may vary within $[10^{-1}, 10^2]$keV, the stellar radius to be equal to 10km, the parameter m assumes values in the interval $[5.5\times10^{-11}, 1.7\times10^{-5}]M_\odot$.

The parameter ℓ given by Eq.(41), then, is related to the parameters of the model as follows

$$\ell \simeq 2.3 \left(\frac{B}{10^9 \rm G}\right)^{\frac{2}{3}} \left(\frac{K_{\rm B}T}{\rm keV}\right)^{\frac{1}{6}} \left(\frac{M_{\rm d}}{M_\odot}\right)^{-\frac{1}{3}} {\rm cm}\,. \tag{45}$$

For different values of the mass of the accretion disk $M_{\rm d}$ and for a B-field of 10^{12} Gauss, the behavior of ℓ and the ratio $\lambda/L_{\rm s}$ (40) as functions of the temperature are given in Fig.5 and Fig.6 respectively. From the behavior of ℓ, it is evident that the condition $\ell \ll R_{\rm S}$ is always verified.

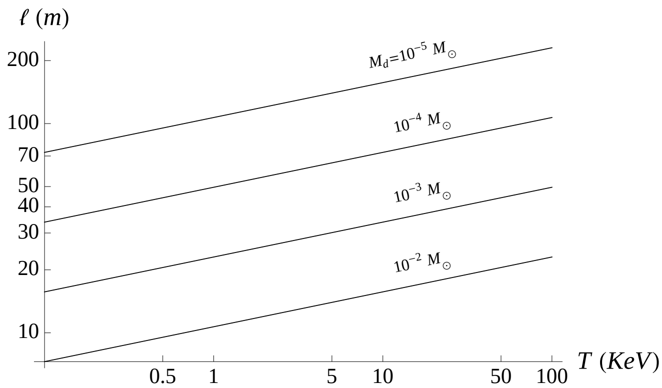

Figure 5. Behavior of the characteristic length ℓ as a function of the temperature T of the disk. The different values above each single plot denotes the mass of the accretion disk $M_{\rm d}$ in Solar mass units for which Eq.(45) is evaluated. It is worth noting how such a characteristic length is always much smaller than the typical dimensions of a disk.

[2] Such fraction is roughly proportional to the ratio $L_{\rm s}/R_{\rm ext}$, where $R_{\rm ext}$ denotes the external radius of the disk configuration.

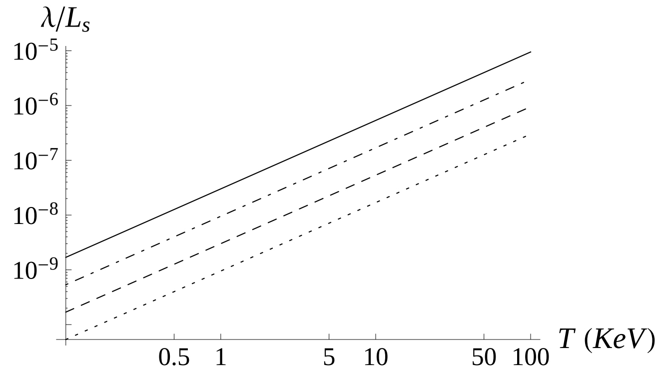

Figure 6. Plot of the λ/L_s ratio as a function of the temperature T of the disk. The different curves correspond to different values of $M_d/M_\odot$ (from top to bottom: $10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}$) for which Eq.(40) is evaluated.

VII. EXTREME NON-LINEAR REGIME

We now analyze the opposite case with respect to the linear regime, when the electromagnetic backreaction dominates the background dipole contribution (i.e., $\partial_r\zeta \gg \partial_r\psi_D$) and the perturbed mass density $\hat{\rho}$ is larger than $\bar{\rho}$ fixed by the gravostatic equilibrium (33); more precisely, we can apply the following hierarchy relations

$$\frac{\zeta}{\psi_{D0}} \gg \frac{1}{kr}\,, \qquad \hat{\rho} \gg \bar{\rho}\,. \tag{46}$$

Therefore, we can also require that the gravitational term of the system (31) be negligible with respect to the pressure gradient, i.e., (here we are assuming $\hat{p} = \hat{v}_s^2\hat{\rho}$)

$$\frac{r}{\hat{L}_s} \gg \frac{2hz}{r^2}\,, \tag{47}$$

where $\hat{L}_s \equiv GM_*/2\hat{v}_s^2$. We now split the pressure term in analogy to the surface function ζ, i.e., $\hat{p}(r,z^2) = \hat{q}(r,z^2)/r$, so that

$$\partial_r\hat{p} = \frac{1}{r}\left(\partial_r\hat{q} - \frac{1}{r}\hat{q}\right) \simeq \frac{1}{r}\partial_r\hat{q}\,, \tag{48}$$

valid for $kr \gg 1$. Hence, fixing the mass density on the equatorial plane in the form $\rho_0 = m'/\sqrt{r}$, the configuration system (31) reads as

$$\frac{3GM_*m'}{\mu_0}\hat{D}\phi = \partial_r\hat{q} + \frac{1}{4\pi}\partial_r\phi\Delta\phi\,, \tag{49a}$$

$$\partial_z\hat{q} + \frac{1}{4\pi}\partial_z\phi\Delta\phi = 0\,. \tag{49b}$$

In order to analyze the system above, let us introduce the following dimensionless quantities

$$\begin{gathered} x = \tilde{k}r\,, \qquad u = \frac{z}{h}\,, \qquad \xi = \frac{1}{(\tilde{k}h)^2}\,, \\ S = \frac{\hat{q}}{\mu_0^2\tilde{k}^5}\,, \qquad \tilde{k} = \left(\frac{3GM_*m}{\mu_0^2L^{5/2}}\right)^{2/9}\,, \\ \Phi = -\frac{\phi}{\mu_0\tilde{k}^{3/2}}\,, \end{gathered} \tag{50}$$

where we set $m' = \tilde{m}/L^{5/2}$, with $L = \text{const}$. By such definitions, system (49) rewrites in the form

$$\hat{D}\Phi + \partial_x S + \frac{1}{4\pi}\partial_x\Phi\left(\partial_x^2\Phi + \xi\partial_u^2\Phi\right) = 0\,, \tag{51a}$$

$$\partial_u S + \frac{1}{4\pi}\partial_u\Phi\left(\partial_x^2\Phi + \xi\partial_u^2\Phi\right) = 0\,. \tag{51b}$$

We now solve these two equations in close analogy to what is done in [2] in this same extreme non-linear regime. We consider a function Φ in the form $\Phi(x,u^2) = N(x)F(u^2)$, where N is an odd function of the radial variable. In the limit $\xi \to 0$, which we are focusing on, from Eq.(51b), we easily get

$$S(x,u^2) = -\frac{1}{8\pi}F^2N\frac{d^2N}{dx^2}\,. \tag{52}$$

Substituting this expression into Eq.(51a) and setting $\hat{D}(x,u) = K(x)F(u^2)$, we eventually get

$$8\pi K(x) = \frac{d^3N}{dx^3} - \frac{1}{N}\frac{dN}{dx}\frac{d^2N}{dx^2}\,, \tag{53}$$

where we must necessarily require $K \geq 0$, to ensure the positivity of the mass density $\hat{\rho} = \hat{D}\rho_0$.

If we assume the following simple form for $N(x)$

$$N(x) = A\left[\sin(x) + B\sin(2x)\right]\,, \tag{54}$$

then

$$8\pi K(x) = \frac{6AB\sin^2(x)}{2B\cos(x)+1}\,. \tag{55}$$

Finally, we can calculate the quantity $\hat{v}_s^2$ via the relation

$$\begin{aligned} \hat{v}_s^2 &= \frac{\hat{p}}{\hat{\rho}} = \frac{\mu_0^2k^{11/2}L^{5/2}}{m}\times \\ &\times\frac{AF\left(u^2\right)\left[2B\cos(x)+1\right]^2\left[8B\cos(x)+1\right]}{6B\sqrt{x}}\,. \end{aligned} \tag{56}$$

Since we have to require $K(x)$ and $\hat{v}_s^2$ to be positive definite, then parameters A and B must satisfy the following conditions

$$A > 0\,, \qquad 0 < B \leq \frac{1}{8}\,. \tag{57}$$

For the particular choice $A = 1$ and $B = 1/8$, and $F(u^2) = 1$, the (effective) perturbed sound velocity $\hat{v}_s^2$ (56) and the perturbed mass density

$$\hat{\rho} = \hat{D}\rho_0 = \frac{K(x)\tilde{m}}{L^{5/2}\sqrt{r}} = \frac{3\tilde{m}\sin^2(x)}{8\pi L^{5/2}\sqrt{x/k}\,[\cos(x)+4]}\,, \quad (58)$$

are depicted in Fig.7. The radial dependence of $\hat{v}_s^2$ can be interpreted as a corresponding behavior for the temperature perturbation induced by the plasma back-reaction. For sufficiently small scales of the crystalline structure, this temperature contribution becomes the dominant one and, in this scheme, the disk acquires a non-isothermal profile.

We stress how, even in this global model, the emergence of a ring-sequence decomposition of the disk takes place as far as the induced magnetic field dominates the central object one, i.e., when the perturbations have a sufficiently small scale such that $\partial_r\zeta \gg \partial_r\psi_{\mathrm{D}}$. This fact confirms that a thin disk admits a plasma configuration characterized by microstructures which have the form of double and opposite current filaments. The instability properties of such radial profile of the disk is expected to deeply influence the averaged transport features of the global structure, especially in view of energy and angular momentum transfer from the micro- to the mesoscales of the system.

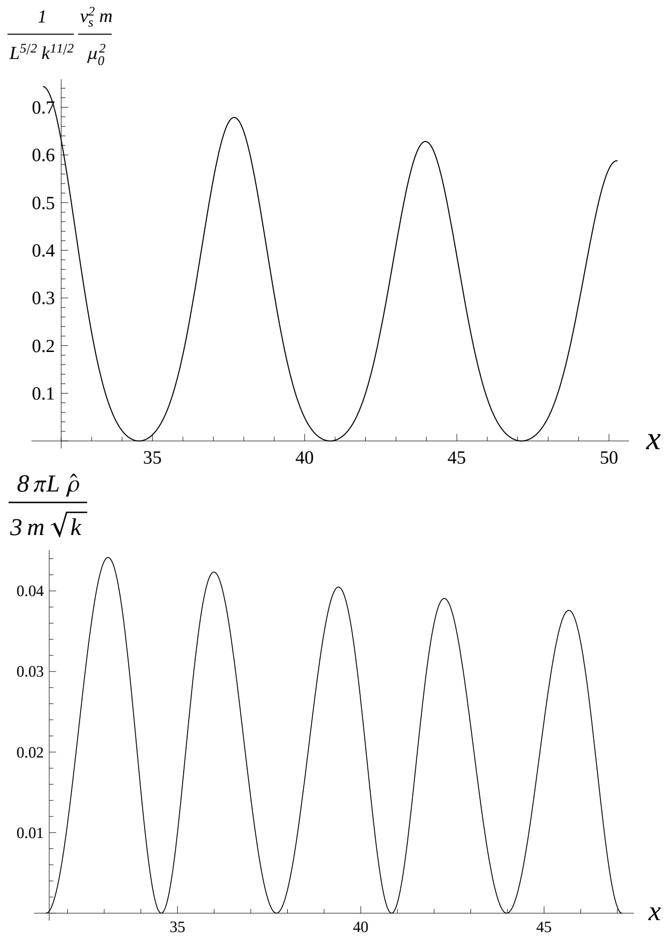

Figure 7. Adimensional sound velocity $\hat{v}_s^2$ (top) and perturbed mass density $\hat{\rho}$ (bottom) for the case $A = 1$ and $B = 1/8$, and $F(u^2) = 1$.

VIII. CONCLUDING REMARKS

We considered a two-dimensional axisymmetric ideal MHD model to describe the structure of a plasma disk surrounding a compact astrophysical object. We neglected the self-gravity of the disk and treated the central gravitational field as a Newtonian profile induced by the mass of the compact body. The magnetic field external to the plasma is described by an exact dipole-like structure associated with the intrinsic features of the central astrophysical source. We then settled the equilibrium configuration of the plasma by accounting for the internal electromagnetic back-reaction, described via static perturbations of the magnetic flux surfaces and of the thermodynamical quantities. This way, we fixed the global model associated with the analysis pursued in [1, 2], where a crystalline profile and a ring sequence for the disk emerged as the result of local configurations confined around fixed values of the radial coordinate. In this respect, we neglected the poloidal component of the matter flux, retaining the disk rotation as the only dominant effect. Then the drift ordering was taken into account when splitting the configuration variables into background and perturbation components. The possibility to link the present global model to the local configuration is offered by a suitable radial scaling of the amplitudes of the oscillating-like behaviors emerging at a fixed radius; furthermore our scheme relies on the small scale nature of the perturbation in comparison to the disk size. We were able both in the linear and in the extreme non-linear cases, discussed above in detail, to recover the corresponding local equilibrium profiles. The amplitude scaling of the perturbed magnetic flux surfaces, mass density and pressure were properly fixed. These results show that the crystal structure of the magnetic field and the ring-like profile of the mass density are rather general features of the two-dimensional, axisymmetric, ideal MHD model.

The derivation of the radial scaling requires, however, a certain number of restrictions on the fundamental quantities involved in setting the plasma distribution. A key request is that the plasma (suffering this decomposition in periodic substructures) must be confined within a certain region from the central object, much smaller than the characteristic length L_{s} defined in Eq.(36). In the case of the most common stellar sources, this request suggests that the crystal morphology of the disk be preferably expected in relative low-temperature plasma disks, while the case of a very hot disk profile seems to require

a very large value of the central mass. For instance, in the case of plasma structures surrounding the very massive black holes at the center of Active Galactic Nuclei, the central mass is typically of the order $10^8 - 10^9$ Solar masses, leading to very high values of the length L_s. However both in the stellar and in the galactic context, the internal nature of the crystal profile, located sufficiently close to the central object, suggests the necessity for General Relativistic effects in describing the gravitational interaction. For a discussion of the role of the present configuration of plasma and its instabilities in the Active Galactic Nuclei morphology and for the necessity to include relativistic corrections, see [18]. Finally we stress how in Sec. VI, we outlined that such restrictions on the radial region where the crystalline structure may take place is at all equivalent to requesting the thinness of the disk. In this sense, it is immediate to recognize the class of accretion disks to which the obtained result can apply, i.e., all those configurations for which the sound velocity is much smaller than the rotational flow of the disk.

This work was developed within the framework of the CGW Collaboration (http:// www.cgwcollaboration.it). R.B. acknowledges ICRA (International Center for Relativistic Astrophysics) for having partially supported this work.

[1] B. Coppi, Phys. Plasmas **12**, 7302 (2005).
[2] B. Coppi and F. Rousseau, Astrophys. J. **641**, 458 (2006).
[3] G. S. Bisnovatyi-Kogan and R. V. E. Lovelace, New Astron. Rev. **45**, 663 (2001).
[4] T. Piran, Phys. Rep. **314**, 575 (1999).
[5] J. H. Krolik, *Active galactic nuclei: from the central black hole to the galactic environment* (Princeton Univ. Press, 1999).
[6] D. Lynden-Bell, Mon. Not. R. Astron. Soc **279**, 389 (1996).
[7] H. Spruit, Lect.Notes Phys. **794**, 233 (2010).
[8] E. Velikhov, Sov. Phys. JETP **36**, 995 (1959).
[9] S. Chandrasekhar, Proceedings of the National Academy of Science **46**, 253 (1960).
[10] S. A. Balbus and J. F. Hawley, Astrophys. J. **376**, 214 (1991).
[11] S. A. Balbus and J. F. Hawley, Rev. Mod. Phys. **70**, 1 (1998).
[12] N. I. Shakura, Soviet Astron. **16**, 756 (1973).
[13] N. I. Shakura and R. A. Sunyaev, A.&A. **24**, 337 (1973).
[14] B. Coppi, Plasma Phys. Contrl. Fus. **36**, B107 (1994).
[15] G. I. Ogilvie, Mon. Not. R. Astron. Soc **288**, 63 (1997).
[16] B. Coppi, Proceedings of the 2007 Conference on Plasma Physics of the European Physical Society (2007).
[17] M. Lattanzi and G. Montani, Europhys. Lett. **89**, 39001 (2010).
[18] B. Coppi, Plasm. Phys. Contr. Fus. **51**, 124007 (2009).
[19] G. Montani and N. Carlevaro, Phys. Rev. E **82**, 025402 (2010).
[20] B. Coppi, Europhys. Lett. **82**, 19001 (2008).
[21] G. S. Bisnovatyi-Kogan and R. V. E. Lovelace, Astrophys. J. **529**, 978 (2000), arXiv:astro-ph/9902344.
[22] B. Coppi and P. S. Coppi, Phys. Rev. Lett. **87**, 051101 (2001).
[23] B. Coppi, Phys. Rev. Lett. **39**, 939 (1977).
[24] D. Dobrott, D. B. Nelson, J. M. Greene, A. H. Glasser, M. S. Chance, and E. A. Frieman, Phys. Rev. Lett. **39**, 943 (1977).
[25] G. Montani and R. Benini, Gen. Rel. Grav. **43**, 1121 (2011), 10.1007/s10714-010-1038-9.
[26] G. Montani and N. Carlevaro, Phys. Rev. E **82**, 025402 (2010), arXiv:1007.5401 [astro-ph.HE].
[27] V. C. A. Ferraro, Mon. Not. R. Astron. Soc **97**, 458 (1937).